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Intersecting hypersurfaces, topological densities and Lovelock gravity

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Abstract

Intersecting hypersurfaces in classical Lovelock gravity are studied exploiting the description of the Lovelock Lagrangian as a sum of dimensionally continued Euler densities. We wish to present an interesting geometrical approach to the problem. The analysis allows us to deal most efficiently with the division of spacetime into a honeycomb network of cells produced by an arbitrary arrangement of membranes of matter. We write the gravitational action as bulk terms plus integrals over each lower dimensional intersection.

The spin connection is discontinuous at the shared boundaries of the cells, which are spaces of various dimensionalities. That means that at each intersection there are more than one spin connections.

We introduce a multi-parameter family of connections which interpolate between the different connections at each intersection. The parameters live naturally on a simplex. We can then write the action including all the intersection terms in a simple way. The Lagrangian of Lovelock gravity is generalized so as to live on the simplices as well. Each intersection term of the action is then obtained as an integral over an appropriate simplex.

Lovelock gravity and the associated topological (Euler) density are used as an example of a more general formulation. In this example one finds that singular sources up to a certain co-dimensionality naturally carry matter without introducing conical or other singularities in spacetime geometry.

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1. Introduction

In the light of the current trends in high energy physics, it is widely supposed that spacetime has dimensions higher than four. In studying classical gravity, there are then other terms in the gravitational action, yielding second order field equations, which it is reasonable to consider. In d dimensions we have the general Lagrangian, first obtained by

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Lovelock [1]. We use the vielbein formulation [2,3]:

$$\mathcal{L} = \sum_{n=0}^{[d/2]} \alpha_n f(\Omega^{\wedge n} \wedge E^{\wedge d-2n}), \tag{1}$$

$$f(\Omega^{\wedge n} \wedge E^{\wedge d-2n}) = \Omega^{a_1 a_2} \wedge \cdots \wedge \Omega^{a_{2n-1} a_{2n}} \wedge E^{a_{2n+1}} \wedge \cdots \wedge E^{a_d} \epsilon_{a_1 \dots a_d}.$$

Above, E^a are the vielbein frames and Ω^{ab} is the curvature 2-form:

$$\Omega^{ab} = \mathrm{d}\omega^{ab} + \frac{1}{2}[\omega, \omega]^{ab} = \mathrm{d}\omega^{ab} + \omega^a_c \wedge \omega^{cb}.$$

The spin connection is ω. The Lie bracket for a p-form A and q-form B is $[A, B]^{ab} = A^a_c \wedge B^{cb} - (-1)^{pq} B^a_c \wedge A^{cb}$. The totally antisymmetric tensor is normalized to $\epsilon_{01\dots d} = +1$. $[d/2]$ is the highest integer less than $d/2$. There are $[d/2]$ coefficients α_n.

The first term in (1) is the cosmological constant. The second term is the Einstein–Hilbert Lagrangian. In the familiar four dimensions it reads:

$$\mathcal{L}_{EH} = \alpha_1 \Omega^{ab} \wedge E^c \wedge E^d \epsilon_{abcd} = 2\alpha_1 R\sqrt{-g}$$

and the coefficient in our conventions is related to Newton's constant by $\alpha_1 = (8\pi G)^{-1}$. In dimensions higher than four, there are other terms in (1) which are corrections to the Einstein theory. Each term is a polynomial of order n in the curvature. These were studied in the late 1980's when it was realized that they were related to strings and were ghost free in a flat background [3,4]. The relation to strings was further discussed [5–7], exact solutions were obtained and black hole spacetimes were analyzed in various dimensions [8–16]. Recently the special properties of the theory have been studied motivated by braneworld models [17,18], higher dimensional black holes and also Chern–Simons Gravity [19]. To us, the important property of the Lagrangian (1) is its affinity to topological densities. It is the fact that will enable us to deal with the problem of intersecting hypersurfaces filled with matter in this theory with ease and generality. It will also allow us to write the solution in an elegant form.

In $d = 2n$, the term $f(\Omega^{\wedge n})$ is proportional to the Euler density and is locally a total derivative. For example if we write the Einstein–Hilbert term in two dimensions it reads

$$\alpha_1 \Omega^{ab} \epsilon_{ab}$$

(the coupling α_1 has dimension zero). Integrated over a closed manifold M it just gives the Euler number of the manifold. No local information can be obtained from this. The analogous quantity in arbitrary even dimension is

$$\alpha_n \ \Omega^{a_1 a_2} \dots \Omega^{a_{2n-1} a_{2n}} \epsilon_{a_1 \dots a_{2n}}.$$

This quantity is equally mute about local information. By analogy, in $d > 2n$, the quantity $f(\Omega^{\wedge n} \wedge E^{d-2n})$ is known as the dimensionally continued Euler density [3,20,21]. When written in terms of tensors, the dimensionally continued density takes the same form as the Euler density,

$$\frac{1}{2^n} \ \delta^{\mu_1 \dots \mu_{2n}}_{\nu_1 \dots \nu_{2n}} R^{\nu_1 \nu_2}{}_{\mu_1 \mu_2} \cdots R^{\nu_{2n-1} \nu_{2n}}{}_{\mu_{2n-1} \mu_{2n}} \sqrt{-g} \tag{2}$$

except that the dummy indices run over more values. In the vielbein notation, the difference is more clear — the vielbeins appear explicitly in the dimensionally continued density. The Lagrangian formulation of Lovelock gravity involves a sum of terms which are dimensionally continued Euler densities and yields the Lovelock equations of motion [1]. This is a useful way to think of the Lagrangian. This similarity to the Euler density accounts for the interesting properties of the theory mentioned in the previous paragraph.

In this paper, we deal with singular sources of gravity, that is matter whose internal structure is restricted in dimensionality lower than of that of the manifold. It is known that, of all singular sources in Einstein's theory, the codimension 1 source [22,23] is especially easy to describe mathematically. The stress–energy–momentum tensor is unambiguously well defined as a distribution. It has recently been realized that this is also true for codimension 1 hypersurfaces in Lovelock theory [18]. It is also known that, due to the non-linearity of Einstein's theory, there are problems and ambiguities in describing singular sources of codimension greater than 1 [23], although there is some hope of being able to describe codimension 2 sources in a meaningful way [24,25]. Just as point charges are useful in studying electromagnetism, it is also useful to have a well defined description of singular sources of gravity. Even if singular sources do not exist as fundamental particles, they can be useful as simple approximations.

Hypersurfaces of codimension one (hereafter just called hypersurfaces), will generally intersect each other. It is then a natural step to consider intersections. In Ref. [26] we found that there could be a singular energy–momentum tensor located at intersections without any mathematical problems or ambiguities (in particular, the vielbein frame is well defined at the intersections). In the order n Lovelock gravity, the singular matter can live on intersections of codimension n or less. Some examples of intersections in Lovelock gravity have also been given in Refs. [27].

Before going into the details of intersections, let us first consider a hypersurface and the junction conditions [22]. At a junction, the metric is continuous but the normal derivative jumps. The part of the curvature that is intrinsic to the junction is single valued but the extrinsic curvature representing the embedding of the surface into the manifold is different on each side. In the vielbein language it is the connection 1-form that is discontinuous. The problem then is that there are discontinuous forms meeting at intersections. We would like to re-express the problem in terms of continuous connection 1-forms so as to use usual methods of differential geometry. To give a specific example, consider the so called Einstein–Gauss–Bonnet theory in five dimensions. There are three terms: the cosmological constant, Einstein–Hilbert and a curvature squared term:

$$\mathcal{L} = \alpha_0 E^a \wedge E^b \wedge E^c \wedge E^d \wedge E^e \epsilon_{abcde} + \alpha_1 \Omega^{ab} \wedge E^c \wedge E^d \wedge E^e \epsilon_{abcde} + \alpha_2 \Omega^{ab} \wedge \Omega^{cd} \wedge E^e \epsilon_{abcde}.$$

Suppose that we have a single hypersurface Σ which divides our spacetime into two regions M_- and M_+, such that the metric is continuous but the connection may be discontinuous. The junction conditions can be obtained by including the surface term in the action:

$$\begin{aligned}&\alpha_1 \int_\Sigma (\omega_+^{ab} - \omega_-^{ab}) \wedge E^c \wedge E^d \wedge E^e \epsilon_{abcde} \\ &\quad + \alpha_2 \int_\Sigma (\omega_+^{ab} - \omega_-^{ab}) \wedge \left(\Omega_+^{cd} + \Omega_-^{cd} - \frac{1}{3} (\omega_+ - \omega_-)^c{}_f \wedge (\omega_+ - \omega_-)^{fd} \right) \wedge E^e \epsilon_{abcde}. \end{aligned} \tag{3}$$

The Euler–Lagrange variation with respect to the connection cancels the total derivative term coming from the bulk. The Euler variation with respect to the vielbein gives a tensor which, when set equal to the intrinsic stress tensor on Σ, gives the correct junction conditions [28,29]. It is usual to introduce the intrinsic connection $\omega_\parallel$, and the corresponding curvature $\Omega_\parallel$ and second fundamental form $\theta^{ab} := \omega^{ab} - \omega_\parallel^{ab}$. Then the surface term would have the form:

$$\int_\Sigma \alpha_1 \left[\theta^{ab} \wedge E^c \wedge E^d \wedge E^e \epsilon_{abcde} \right]_-^+ + 2\alpha_2 \left[\theta^{ab} \wedge \left(\Omega_\parallel^{cd} + \frac{1}{3} \theta^c{}_f \wedge \theta^{fd} \right) \wedge E^e \epsilon_{abcde} \right]_-^+$$

where the brackets signify the jump in this surface term across the boundary: $[f(\theta)]_-^+ := f(\theta_+) - f(\theta_-)$. The first term is the jump in the Gibbons–Hawking boundary term (first written, it seems, by York [30]), written in terms of differential forms. The second term is the jump in the boundary term of Myers [31]. It is of course natural that the boundary term which makes the action well defined on a manifold with boundary also plays a role in the junction conditions.

The advantage of introducing the intrinsic connection into the action is that one can write everything in terms of extrinsic and intrinsic curvature tensors on Σ. One can then evaluate the junction conditions using bulk metrics written in different coordinate systems on each side. The intrinsic curvature is then written in terms of the intrinsic reference frame on Σ. Then the difference is taken between the two terms. In this way one can avoid the dangers of confusing a real discontinuity with a purely coordinate effect [22]. This approach is important if one is calculating in terms of tensors, but not so essential if one uses differential forms, a point which will be discussed in more detail at the end of Section 5. In the following we shall introduce surface terms which, like (3), do not explicitly contain the induced connections on the hypersurfaces. In Section 6 we will return to make contact with the formulation in terms of extrinsic curvature.

The question arises, can one describe intersections of hypersurfaces in the same style, with boundary terms? The key point in Ref. [26] is that this can be done, thanks to the relationship of each term in the Lagrangian to its topological cousin, the Euler density.

Now the Euler number is something that is actually independent of the local form of the metric and associated connection,

$$\int_{M^{(2n)}} f(\Omega^{\wedge n}) = \int_{M^{(2n)}} f((\Omega')^{\wedge n}) \;\propto \text{Euler no.} \tag{4}$$

It is a purely topological number. If we have a whole family of (metric respecting) connections over M, ω_i, related to each other by homotopy, one can equally well write the Euler number in terms of any of them. Also, and the important point for us, one can formally rewrite the Euler number in terms of a discontinuous connection, which coincides with each ω_i in some region i of M. One will then have boundary and intersection terms in the integral. This amounts to a cellular decomposition of the manifold into a honeycomb-type lattice. Note, we do not localize curvature at the intersections (which would introduce the complication of the connection picking up a gauge transformation going round the singularity).

The set of boundary and intersection terms were found in the previous work and are summarized in Section 2. We introduced a connection which interpolated between each ω_i by means of some variables usually denoted by t which we shall call homotopy parameters. We found that each intersection term was a density built from the curvature of the interpolating connection, integrated over the homotopy parameters. In Section 3, we shall re-derive these results by a more geometrical method. We introduce a manifold, W, which is locally a Cartesian product of each intersection and a simplex in the homotopy parameters. We then introduce a closed form η in the space W. The closure of η implies our composition rule (8), in other words the closure of η is sufficient to provide the composition rule. The results can be presented in a simpler way by introducing a multi-parameter generalization of the Cartan homotopy operator. We should note that these results are essentially the same as results found by Gabrielov et al. in seeking a combinatorial formula for Characteristic Classes [32].

The entire honeycomb formed out of the complicatedly intersecting hypersurfaces is described by a few simple equations. All sorts of intersections which it contains are accommodated in the scheme given by these equations and the shape of W. For the Euler density, we find an explicit expression for η and show that it is closed. The form of the intersection terms is clarified greatly.

In Section 4 we turn to the action built from the dimensionally continued Euler densities, where the vielbein enters explicitly into the action. If there are hypersurface sources, the connection 1-form is discontinuous. Can we still rewrite the action in terms of the continuous connection in each bulk region plus boundary terms? We will show that the answer is yes and that the gravitational intersection Lagrangians obey the same composition rule (8). This is because the dimensionally continued η is still closed on W.

In Ref. [26], we generalized the intersection terms to the dimensionally continued Euler densities in a natural way. The resulting action was found to be one-and-a-half order in the connection: the field equations come from independent variation of vielbein and connection. The zero torsion constraint makes the field equation from the variation of the connection vanish identically. We thus concluded that this was the correct action, the explicit variation with respect to the vielbein giving the junction conditions for intersecting hypersurfaces in Lovelock gravity. The key results of Section 5, Propositions 6–9, verify this.

We can write the action which generates all the intersection terms as

$$S = \int_W \eta \tag{5}$$

η is given by (17) for the Euler density and (51) for the dimensionally continued Euler density. The validity of the formula in both the topological and the gravitational case is the main result of this paper.

In Section 6 the junction conditions for intersections and collisions are elaborated in more detail and some physical applications are discussed.

2. The composition rule

We will review the argument of Ref. [26]. Let ω be any connection and Ω the curvature. The continuous variation of an invariant polynomial

$$P(\Omega) = f(\Omega^{\wedge n}) \tag{6}$$

with respect to the connection produces the well known formula

$$P(\Omega) - P(\Omega') = \mathrm{d}TP(\omega, \omega') \tag{7}$$

where TP is the Transgression of P [41]. This was generalized to the composition formula:

$$\sum_{s=1}^{p} (-1)^{s-p-1} \mathcal{L}(\omega_0, \ldots, \widehat{\omega_s}, \ldots, \omega_p) = \mathrm{d}\mathcal{L}(\omega_0, \ldots, \omega_p). \tag{8}$$

$\mathcal{L}(\omega) = P(\Omega)$, $\mathcal{L}(\omega_1, \omega_2) = TP(\omega_1, \omega_2)$ and an expression for the general $\mathcal{L}$ was found. It was shown that these forms live on the intersections of regions of M. Let's divide M up into a honeycomb of regions labelled by i and denoted by $\{i\}$, with intersections denoted by $\{ij\}$, $\{ijk\}$ etc, which are symbols fully antisymmetric in their indices and keep track of the orientation. The integral over the manifold of $\mathcal{L}(\omega)$ can be rewritten

$$\int_M \mathcal{L}(\omega) = \sum_i \int_{\{i\}} \mathcal{L}(\omega_i) + \sum_{k \geq 2} \frac{1}{k!} \sum_{i_1 \ldots i_k} \int_{\{i_1 \ldots i_k\}} \mathcal{L}(\omega_{i_1}, \ldots, \omega_{i_k}). \tag{9}$$

Explicit formulae for these intersection terms were found. Each intersection contributes to the action a term:

$$\int \mathrm{d}^{d-p} x \int \mathrm{d}^p t \,\mathrm{funct.}(\omega(t)). \tag{10}$$

where t are the homotopy parameters and $\omega(t)$ interpolates between the ω_i's. We will find somewhat simpler expressions for these terms in the next section.

This composition rule applies to any invariant polynomial, such as the Pontryagin Class. Because of our interest in Lovelock gravity, we shall only discuss here the Euler density. The connection ω is always the Lorentzian (or Riemannian) connection and torsion free.

In the dimensionally continued case the left hand side of (9) with the $\mathcal{L}(\omega_{i_1}, \ldots, \omega_{i_k})$ being replaced by the their dimensionally continued analogues, is the gravitational action describing the system involving intersecting hypersurfaces. The variation with respect to the connections gives identically zero under zero torsion and the variation with respect to the vielbein provides the junction conditions. The existence of these intersection Lagrangians, coming from the discontinuities of the connection, shows that, in order to have a general treatment, one should allow for the possibility of distributional parts of the matter's energy tensor with support at the discontinuities

$$T = \sum_i T_i f_i + \sum_{k \geq 2} \frac{1}{k!} \sum_{i_1 \ldots i_k} T_{i_1 \ldots i_k} \delta(\Sigma_{i_1 \ldots i_k}) \tag{11}$$

where T_i is the energy tensor of the region labelled by i and f_i is a function that taken on the value 1 in the respective region and 0 elsewhere; $T_{i_1 \ldots i_k}$ is the energy tensor of the intersection hypersurface $\Sigma_{i_1 \ldots i_k}$, the point-set whose orientation as embedded in $\Sigma_{i_2 \ldots i_k}$, $\Sigma_{i_1 i_3 \ldots i_k}$, etc, has been taken into account by the fully antisymmetric symbol $\{i_1 \ldots i_k\}$. $\delta(\Sigma_{i_1 \ldots i_k})$ is the delta function with support $\Sigma_{i_1 \ldots i_k}$.

Note that lower case Latin indices from the middle of the alphabet label the bulk regions, not to be confused with spacetime indices.

3. A geometrical approach

This and the next section will be devoted to analyzing the purely topological case, since part of our arguments can be understood in this context. It will then help us to see what kind of refinements are needed to pass to the gravitational case.

We want to describe the situation in the vicinity of an intersection of codimension p between different bulk regions. In this vicinity there will also be intersections of lower codimension. At each intersection, we have a meeting of connections ω_i in the different regions. Let us for the moment deal only with simplicial intersections: we define the simplicial intersection of codimension p to be a surface of codimension p where $p+1$ regions meet (Fig. 1(a)). It was found in Ref. [26] that the $\mathcal{L}(\omega_0, \ldots, \omega_p)$ is an integral over p different homotopy parameters interpolating between the connections (see (10)).

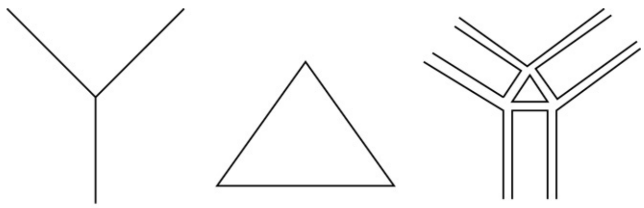

Fig. 1. (a) Simplicial intersection; (b) Simplex in the space of homotopy parameters; (c) A projected diagram of the space W (Eq. (B.6)). Every $d-1$ dimensional surface is 'thickened' in the t-space by a 1-dimensional simplex. These meet at a $d-2$ dimensional intersection, which is "thickened" by a triangle in the t-space.

If we look at (10) we make the following observation: each order of intersection causes us to lose a dimension but gain an extra connection. Each new connection means an extra parameter of continuous variation. With this in mind, we can think of our action as an integral over a d-dimensional space which is a mixture of spacetime and t directions.

Let us interpolate in the most symmetrical way. We introduce a N-dimensional simplex in the Euclidean space $\mathbf{R}^{N+1}$ with coordinates t. Let us define the interpolating connection:

$$\omega(t) := \sum_{i=0}^{N} t^i\, \omega_i, \qquad \sum_{i=0}^{N} t^i = 1 \tag{12}$$

and the associated curvature:

$$\Omega(t) := \mathrm{d}\omega(t) + \frac{1}{2}[\omega(t), \omega(t)]. \tag{13}$$

So we introduce the space $F = S_N \times M$, with S_N, a simplex of dimension N. The latter will be frequently called t-space. Each of N points of the simplex corresponds to a continuous connection form ω_i on M with its support on some open set in M containing the region i; we add another one, that is, one more connection for reasons that will become clear later. Each contribution to our action will live on some d-dimensional subspace of F. The technical reason for introducing the space F is that the connection is continuous on it and integration is well defined. There is also an aesthetic reason. It is quite a nice feature of the problem that the mathematics will take on its simplest form when the t-space is a simplex, as we have already chosen. It provides a geometrical picture which can simplify many calculations. For example, the treatment of a non-simplicial intersection becomes easy, as we shall see in below.

Let us define a d-dimensional differential form in this space F (where for convenience the $\mathrm{d}x$'s are suppressed).

$$\eta := \sum_{l=0}^{n} \frac{1}{l!}\, \mathrm{d}t^{i_1} \wedge \cdots \wedge \mathrm{d}t^{i_l} \wedge \eta_{i_1 \ldots i_l}(x, t), \tag{14}$$
$$\eta_{i_1 \ldots i_l} := \eta_{i_1 \ldots i_l \mu_{l+1} \ldots \mu_d}\, \mathrm{d}x^{\mu_{l+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_d}.$$

We can now proceed to integrate this form over different faces of S_N. A p-face (which we call $s_{0\ldots p}$ or just s) is a subsimplex of S_N which interpolates between a total of $p+1$ different connections (Fig. 1(b)):

$$s_{0\ldots p} = \left\{ (t^0, \ldots, t^p, 0, \ldots, 0) \,\middle|\, t^i \geq 0, \sum_{i=0}^{p} t^i = 1 \right\}. \tag{15}$$

Let us define $\mathcal{L}_{0\ldots p}$ to be the integral over the p-dimensional face[1]:

$$\mathcal{L}_{0\ldots p} := \int_{s_{0\ldots p}} \eta = \frac{1}{p!} \int_{s_{0\ldots p}} \mathrm{d}t^{i_1} \cdots \mathrm{d}t^{i_p}\, \eta_{i_1 \ldots i_p}, \tag{16}$$

[1] Strictly there should be a factor of $(-1)^{P(0,\ldots,p)}$ in the middle term to account for the orientation with respect to S_N. However, we can choose s to have the positive orientation by assuming the points $0 \ldots p$ are in the appropriate order.

η here being understood to be the restriction of η onto s so that the integral is a function of x only. This integral picks out terms in η which are a volume element on the appropriate face. We would like, for an appropriate choice of η, to identify this term with $\mathcal{L}(\omega_0, \ldots, \omega_p)$ as defined in the introduction with respect to the Euler density. We shall see that this can indeed be done and we shall find a simple form for η.

Proposition 1. *A sufficient condition on η such that $\mathcal{L}_{0\ldots p}$ obeys the composition rule (8) is that η be a closed form, $\mathrm{d}_F\eta = 0$. Here the exterior derivative on F is $\mathrm{d}_F = \mathrm{d}_{(x)} + \mathrm{d}_{(t)}$. $\mathrm{d}_{(t)}$ and $\mathrm{d}_{(x)}$ are the exterior derivative restricted to the simplex and to M respectively.*

Proposition 2. *The form of η corresponding to the Euler density is*

$$\eta = f\left([\mathrm{d}_{(t)}\omega(t) + \Omega(t)]^{\wedge n}\right). \tag{17}$$

This formula is already known in the mathematics literature [32]. η is closed on F: $\mathrm{d}_F\eta = 0$.

Proposition 3. *The intersection Lagrangian can be recovered by the specific choice:*

$$\eta_{1\ldots p} = A_p f(\chi_1 \wedge \cdots \wedge \chi_p \wedge \Omega(t)^{\wedge(n-p)}), \tag{18}$$

$$A_p = (-1)^{p(p-1)/2}\frac{n!}{(n-p)!}$$
$$\Rightarrow \mathcal{L}(\omega_0, \ldots, \omega_p) = A_p \int_{s_{01\ldots p}} \mathrm{d}^p t f(\chi_1 \wedge \cdots \wedge \chi_p \wedge \Omega(t)^{\wedge(n-p)}). \tag{19}$$

$\chi_i \equiv \omega_i - \omega_0$ and $\omega(t) = \omega_0 + \sum_{i=1}^{p} t^i \chi_i$. The intersection Lagrangian is zero for $p > n$.

To prove the first proposition, we will need to use Stokes' Theorem on the face s.

$$\int_s \mathrm{d}_{(t)}\eta = \int_{\partial s} \eta. \tag{20}$$

The boundary of the simplex $s_{0\ldots p}$ is

$$\partial s_{0\ldots p} = \sum_{i=0}^{p} (-1)^i s_{0\ldots\hat{i}\ldots p} \tag{21}$$

with the orientation being understood from the order of the indices.

Now let us integrate the form $\mathrm{d}_{(x)}\eta$ over the face. We will need to remember that in permuting this exterior derivative past the $\mathrm{d}t$'s we will pick up a $\pm$ factor.

$$\mathrm{d}_{(x)}\eta = \sum_l \frac{(-1)^l}{l!} \mathrm{d}t^{i_1} \wedge \cdots \wedge \mathrm{d}t^{i_l} \wedge \mathrm{d}_{(x)}\eta_{i_1\ldots i_l}. \tag{22}$$

Using this information we may integrate over the p-face s

$$\int_s \mathrm{d}_{(x)}\eta = (-1)^p \mathrm{d} \int_s \eta. \tag{23}$$

Combining Eqs. (20) and (23)

$$\int_{s_{0\ldots p}} \mathrm{d}_F\eta = (-1)^p \mathrm{d} \int_{s_{0\ldots p}} \eta + \sum_{i=0}^{p} (-1)^i \int_{s_{0\ldots\hat{i}\ldots p}} \eta. \tag{24}$$

If our form η is closed in F, $\mathrm{d}_F\eta$ must necessarily vanish term by term in the $\mathrm{d}t$'s and $\mathrm{d}x$'s. The integral on the right hand side of (24) must therefore vanish. Recalling the definition (16) we have proved Proposition 1:

$$\mathrm{d}_F\eta = 0 \Rightarrow \mathrm{d}\mathcal{L}_{0\ldots p} = \sum_{i=0}^{p} (-1)^{p-i-1} \mathcal{L}_{0\ldots\hat{i}\ldots p}. \tag{25}$$

The condition that η be closed indeed implies our composition formula.

The proof of Propositions 2 and 3 is in the appendix. We now turn to discuss the space where the form η lives and write down the topological invariant in the presence of connection discontinuities.

4. *W* space and the action of the system

We have seen that the simplicial intersection is related to a simplex in the parameter space. The simplex with $p+1$ vertices corresponds to a codimension p intersection with $p+1$ bulk regions meeting. The example $p=2$ is shown in Fig. 1(a) — the intersection looks like the simplex turned inside out. As pointed out already, the connection $\omega(t)$ is smooth on F where the d-dimensional Lagrangian density η is defined.

Now define a d-dimensional space $W \subset F$ as follows. Consider a simplicial intersection and set

$$W = \sum_{p=0}^{h} \sum_{i_0 \dots i_p} \frac{1}{(p+1)!} s_{i_0 \dots i_p} \times \{i_0 \dots i_p\} \subset F \tag{26}$$

where h is the highest codimension of the intersections present. Let's write the first few terms explicitly

$$W = \sum_i s_i \times \{i\} + \sum_{i<j} s_{ij} \times \{ij\} + \sum_{i<j<k} s_{ijk} \times \{ijk\} + \cdots . \tag{27}$$

The first term contains the bulk regions multiplied with 0-simplices, the second contains the hypersurfaces multiplied with 1-simplices, the third contains the codimension 2 intersections multiplied with the associated 2-simplex, and so on. All the intersection submanifolds of M have been put in a single d-dimensional space, where again $d = \dim M$. Moreover this is a d-dimensional space where the connection $\omega(t) = \sum_i t^i \omega_i$ is *continuous*.

Define the curvature associated with the connection $\omega(t)$ over $W \subset F$, that is, the derivative operator is d_F:

$$\Omega_F := \mathrm{d}_F \omega(t) + \frac{1}{2}[\omega(t), \omega(t)]. \tag{28}$$

From that we have

$$\Omega_F = \mathrm{d}_{(t)} \omega(t) + \Omega(t). \tag{29}$$

Being a curvature form it satisfies the Bianchi identity. To see this explicitly let D_F be the covariant derivative associated with the derivative operator d_F and connection $\omega(t)$. We have

$$D_F \Omega_F = (\mathrm{d}_{(t)} + D(t))(\mathrm{d}_{(t)}\omega(t) + \Omega(t)) = \mathrm{d}_{(t)}\Omega(t) + D(t)\mathrm{d}_{(t)}\omega(t) + D(t)\Omega(t) = 0 \tag{30}$$

as the first two terms cancel each other by $\mathrm{d}_{(t)}\mathrm{d}_{(x)} = -\mathrm{d}_{(x)}\mathrm{d}_{(t)}$ and the last by the Bianchi identity of $\Omega(t)$. This is discussed in more detail in Appendix A.

The action takes the simple form on W:

Proposition 4. *The action of the whole system takes the form*

$$S = \int_W \eta, \quad \eta = P(\Omega_F). \tag{31}$$

Proof. It is an immediate consequence the form of action (9) derived in [26] and of the definition of W above under the consistent identification (16) according to Propositions 2 and 3.[2] We return to prove this explicitly in the next section, for the more general case of the dimensionally continued density by the methods introduced in this paper.

The manifold W is further discussed in Appendix B where some discussion of topology is also included. There it is proved the following

[2] It is an abuse of terminology to speak about 'action' in the purely topological case. In the next section it will become clear why the real gravitational action is found in almost the same way.

Proposition 5. *If* $\partial M = 0$ *then*

$$\partial_F W = 0. \tag{32}$$

In fact it is shown as being equivalent to the definition of the simplicial intersection's boundary rule introduced in [26]. This is a homological version of the statements of the previous section. Also this proposition helps us prove easily the invariance of the quantity $\int_W P(\Omega_F)$ under continuous variations of the connection: under $\omega(t) \to \omega(t) + \delta\omega$ we have easily have $\delta\Omega_F = D_F \delta\omega$ so

$$\delta \int_W P(\Omega_F) = \int_W nf(D_F\delta\omega \wedge \Omega_F^{\wedge(n-1)}) = \int_W n\mathrm{d}_F f(\delta\omega \wedge \Omega_F^{\wedge(n-1)}) = 0 \tag{33}$$

where in the second equality we used the invariance of f (see Appendix A) and the Bianchi identity (30) and in the last step the previous proposition. In the next section this will translate to a well defined variational principle of a gravity action, i.e. a functional providing equations of motion for E and ω.

The proof of Proposition 1 can be thought of in terms of a generalization of Cartan homotopy formula to a higher number of homotopy parameters. Let the operator K_s be defined by $K_s\eta := \int_s \eta$ and let $K_{\partial s} := \int_{\partial s} \eta$. The Eq. (24) can be written as

$$(K_s \cdot \mathrm{d}_F - (-1)^m \mathrm{d}_F \cdot K_s)\eta = K_{\partial s}\eta. \tag{34}$$

This reduces to the usual Cartan homotopy formula for the 1-simplex $m = 1$.

$$(K_{01}\mathrm{d}_F + \mathrm{d}_F K_{01})\eta = \eta(1) - \eta(0). \tag{35}$$

In fact, the whole of our analysis can be reduced down to the two Eqs. (31) and (34). In words: the whole intersection Lagrangian is a density in some manifold which is locally a product of spacetime and a simplex. The composition rules are an expression of this higher dimensional Cartan homotopy operator acting on this density.

Now the non-simplicial intersection in M (that is, when $k > p$ regions meet at a codimension p surface) can be treated quite easily. Instead of integrating over a simplex one integrates over a simplicial complex in t-space. More than one face of dimension p in S_N are associated with the same $(d - p)$-surface in M.

Let us consider a simple example. We have four regions, $1, \ldots, 4$, meeting at a codimension 2 intersection $I \subset M$ which is has no boundary (Fig. 2). There are four hypersurfaces $\{12\}$, $\{23\}$, $\{34\}$ and $\{41\}$ meeting at I. Now look for a 2-chain c such that

$$\partial c = s_{12} + s_{23} + s_{34} + s_{41} \tag{36}$$

as the r.h.s. is a 1-cycle, that is a 1-chain annihilated by the boundary operator ∂. One solution to this equation is

$$c = s_{123} + s_{341} \tag{37}$$

which is clearly not unique: (36) tells us that a new chain differing from the old one with a boundary is another solution. If for example we choose a new 2-chain c'

$$c' = c + \partial s_{1234} = s_{234} + s_{124} \tag{38}$$

still $\partial c' = s_{12} + s_{23} + s_{34} + s_{41}$.

Integrating $\mathrm{d}_F\eta = 0$ over the two sides of (36) we have

$$\int_{s_{12}+s_{23}+s_{34}+s_{41}} \eta = \mathrm{d}\int_c \eta$$
$$\Rightarrow \mathcal{L}_{12} + \mathcal{L}_{23} + \mathcal{L}_{34} + \mathcal{L}_{41} = \mathrm{d}\left(\mathcal{L}_{123} + \mathcal{L}_{341}\right). \tag{39}$$

According to what we have seen in [26] building a functional with a well defined variational principle in the presence of intersections, the last equation tells us that the appropriate term for the non-simplicial intersection $I \subset M$ is

$$\int_I \int_c \eta = \int_I \left(\mathcal{L}_{123} + \mathcal{L}_{341}\right). \tag{40}$$

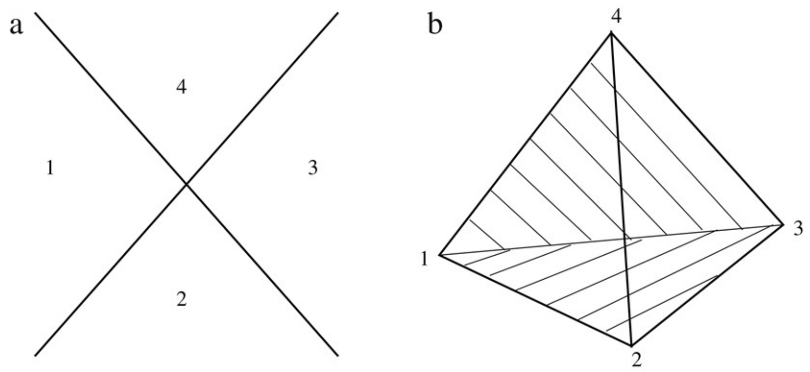

Fig. 2. Non-simplicial Intersection. (a) Spacetime, (b) t-space.

This is a special case of the result obtained in [26] by more conventional means, and can of course be easily checked in our new language: the W space reads

$$W = \sum_{i=1}^{4} s_i \times \{i\} + s_{12} \times \{12\} + s_{23} \times \{23\} + s_{34} \times \{34\} + s_{41} \times \{41\} + c \times I \tag{41}$$

with $\partial\{21\} = \partial\{32\} = \partial\{43\} = \partial\{14\} = -I$ and $\partial I = 0$. Then

$$\partial_F W = (-(s_{12} + s_{23} + s_{34} + s_{41}) + \partial c) \times I \tag{42}$$

which vanishes for any chain c satisfying (36). So, recalling (33), $\int_W \eta$ has a well defined variational principle. The action term at I should be given by $\int_I \int_c \eta$.

This action term is unique: the arbitrariness $c \to c + \partial c'$, for any 3-chain c', does not affect the action as the term above changes by

$$\int_I \int_{\partial c'} \eta = \int_I \int_{c'} \mathrm{d}_{(t)} \eta = -\int_I \int_{c'} \mathrm{d}_{(x)} \eta = \int_{\partial I} \int_{c'} \eta = 0 \tag{43}$$

where we used $0 = \mathrm{d}_F \eta = \mathrm{d}_{(t)} \eta + \mathrm{d}_{(x)} \eta$ in the second equality and $\partial I = 0$ in the last equality and the usual rule of commutation of a t-space integral with $\mathrm{d}_{(x)}$ operator.

5. Dimensionally continued Euler density

So far we have been considering the topological density. This is not much good as a Lagrangian. We know that the action yields no equations of motion. The point is that we can apply what we have learned to the dimensionally continued densities. The Lovelock Lagrangian (1) is a combination of such densities. Now we assume that the connection is a metric compatible (Lorentz) connection. There are now explicit factors of the vielbein frame E^a appearing in the action.

We have a manifold M, of dimension d, with regions labelled by i, separated by surfaces of matter. The vielbein E is continuous but the connection 1-form ω is discontinuous at the surfaces. Once again we rewrite the Lagrangian in terms of the continuous connections ω_i and boundary terms. We interpolate as before:

$$E(t) := \sum_{i=0}^{N-1} t^i \, E_i + t^N E, \qquad \omega(t) := \sum_{i=0}^{N-1} t^i \, \omega_i + t^N \omega \tag{44}$$

and again we define the space $F = S_N \times M$. As well as the d-dimensional manifold W, defined in (26), we also introduce the $(d+1)$-dimensional manifold:

$$W^+ = \sum_{p=0}^{h} \sum_{i_0 \ldots i_p} \frac{1}{(p+1)!} s^+_{i_0 \ldots i_p} \times \{i_0 \ldots i_p\} \subset F, \tag{45}$$

$$s^+_{i_0\dots i_p} = \left\{ (t^0, \dots, t^p, 0, \dots, 0, t^N) \middle| t^i \geq 0, \sum_{i=0}^{p} t^i + t^N = 1 \right\}. \tag{46}$$

The difference from the previous section is that we include the physical vielbein and connection, E, ω, as well as the E_i, ω_i.

We impose the two constraints:

(i) The vielbein frame is continuous across M. At an intersection $\{i_0 \dots i_p\}$:

$$E_{i_0}|_{\{i_0\dots i_p\}} = \dots = E_{i_p}|_{\{i_0\dots i_p\}} = E|_{\{i_0\dots i_p\}}$$

(ii) Each connection is torsion free:

$$\mathrm{d}_{(x)} E^a_i + \omega_i{}^a{}_b \wedge E^b_i = 0.$$

In fact, a good alternative way to define W is: W is the region in F where $E(t, x) = E(x)$ (of course $\mathrm{d}_{(x)} E(t, x)$ is a function of t because the derivative of the metric is discontinuous on M). Let ϕ_+ be the embedding $\phi_+ : W^+ \to F$. Let $D(t)$ be the covariant derivative associated with $\omega(t)$ and $\mathrm{d}_{(x)}$. From the two constraints we derive:

$$\phi^*_+(\mathrm{d}_{(t)} E(t)) = 0, \tag{47}$$

$$\phi^*_+ D(t) E(t)^a = 0. \tag{48}$$

To prove the second equation we have used constraints (i) and (ii) as well as

$$\sum_{i=0}^{p} t^i \mathrm{d}_{(x)} E^a_i + \sum_i t^i \omega_i{}^a{}_b \wedge E(t)^b = \sum_i t^i \omega_i{}^a{}_b \wedge (E(t) - E_i)^b.$$

By the definition of the covariant derivative $D_F = \mathrm{d}_{(t)} + D(t)$, (47) and (48) give

$$\phi^*_+(D_F E(t)) = 0. \tag{49}$$

By these the composition formula is unchanged. To see that this is the case we make use of the invariance property of the polynomial contracted with the epsilon tensor:

$$\phi^*_+ \mathrm{d}_F f\left((\mathrm{d}_{(t)}\omega(t) + \Omega(t))^{\wedge(n)} \wedge E(t)^{\wedge(d-2n)} \right) = \phi^*_+ D_F f\left((\mathrm{d}_{(t)}\omega(t) + \Omega(t))^{\wedge(n)} \wedge E(t)^{(d-2n)} \right) = 0. \tag{50}$$

This vanishes by (28), (30) and (49).

So we can define the form, closed in W^+:

$$\eta_{DC} = f((\mathrm{d}_{(t)}\omega(t) + \Omega(t))^{\wedge(n)} \wedge E(t)^{\wedge(d-2n)}). \tag{51}$$

An alternative notation will be

$$\eta(\omega(t), E(t)) \equiv \eta_{DC}. \tag{52}$$

The intersection terms will be terms in the expansion of η_{DC} integrated over the appropriate simplex in F. We define:

$$\mathcal{L}(\omega_0, \dots, \omega_p, E_0, \dots, E_p) := \int_{s_{0\dots p}} \eta_{DC}.$$

We can now state:

Proposition 6. $\phi^*_+ \mathrm{d}_F \eta_{DC} = 0$.

Further, by (24) the composition rule for $\mathcal{L}(\omega_0, \dots, \omega_p, E_0, \dots, E_p)$ applies, when restricted a codimension $(p-1)$ intersection $\{1 \dots p\}$, where $E_0 = \dots = E_p = E$.

$$\mathrm{d}\mathcal{L}(\omega, \omega_1, \dots, \omega_p, E)|_{\{1\dots p\}} - \sum_{i=1}^{p} (-1)^{p-i-1} \mathcal{L}(\omega, \omega_1, \dots \widehat{\omega_i} \dots \omega_p, E)|_{\{1\dots p\}}$$
$$= (-1)^{p-1} \mathcal{L}(\omega_1, \dots, \omega_p, E)|_{\{1\dots p\}}. \tag{53}$$

The connection ω is the physical (discontinuous) connection. Each term on the left hand side is ill defined, but the sum of them is formally equal to the right hand side.

For the dimensionally continued case, η_{DC} and $\mathcal{L}$ are no longer Euler densities. It was therefore not obvious that our composition formula should survive. It does survive though because η_{DC} is still a closed form when restricted to $W^+ \subset F$.

As a consequence of the composition rule the infinitesimal variation of the action

$$\mathcal{S} = \int_W \eta_{DC} \tag{54}$$

with respect to the connection vanishes [26] (provided we impose the torsion free condition on the connection and continuity of the metric) and the equations of motion just come from the explicit variation with respect to the vielbein.

We can prove this fact now in a neat way using the W space. We assume M has no boundary. Then, according to Appendix B, $\partial_F W = 0$. Under variation $\omega(t) \to \omega(t) + \delta\omega$,

$$\delta_\omega \int_W f(\Omega_F^{\wedge n} \wedge E(t)^{\wedge(d-2n)}) = n \int_W \mathrm{d}_F f(\delta\omega \wedge \Omega_F^{\wedge(n-1)} \wedge E(t)^{\wedge(d-2n)}) = 0 \tag{55}$$

where we have used the invariance of f, the variation of Ω_F: $\delta\Omega_F = D_F\delta\omega$, the identity (30) and the constraint (49) restricted on W. So we have:

Proposition 7. *Under continuity of the vielbein and the torsion free condition on each bulk connection (conditions* (i) *and* (ii) *restricted on W) the field equations for the connection are trivial:* $\delta_\omega \int_W \eta_{DC} = 0$.

We can also show that the various intersection terms in the action do produce a diffeomorphism invariant action functional in the presence of discontinuities. Let ξ be a vector field. Then diffeomorphism invariance of the action $\int_W \eta_{DC}$ can be expressed as

$$0 = \delta_\xi \int_W \eta_{DC} = \int_W \pounds_\xi\, \eta_{DC} \tag{56}$$

where $\pounds$ is the Lie derivative. But

$$\int_W \pounds_\xi\, \eta_{DC} = \int_W i(\xi)\mathrm{d}_F\eta + \mathrm{d}_F i(\xi)\eta_{DC} \tag{57}$$

where we express the Lie derivative via a well known identity involving the inner product operator $i(\xi)$ and used also $i(\xi)\mathrm{d}_{(t)} + \mathrm{d}_{(t)}i(\xi) = 0$ as $\mathrm{d}_{(t)}\xi = 0$. From $\partial_F W = 0$ the second term in the above equation vanishes and we have the condition

$$\int_W i(\xi)\mathrm{d}_F\eta_{DC} = 0 \tag{58}$$

for all vector fields ξ. This implies that

$$\left.\left(\sum_{i=0}^{p}(-1)^{i-p}\mathcal{L}(\omega_0, \ldots, \widehat{\omega_i}, \ldots, \omega_p, E) + \mathrm{d}\mathcal{L}(\omega_0, \ldots, \omega_p, E)\right)\right|_{\{01\ldots p\}} = 0 \tag{59}$$

at an arbitrary intersection chosen here to be $\{01\ldots p\}$. This is a quite different composition rule than (53). Nevertheless it is true, as the quantity on the l.h.s. of (58) does vanish by the conditions (i) and (ii) (and the Bianchi identity (30)).

Let us look again at the non-simplicial intersection. We have seen that the arbitrariness in the choice of the chain c in $\int_I \int_c \eta$ does not affect the action in the purely topological case because $\mathrm{d}_F\eta = 0$. The dimensionally continued density is closed only when restricted to a subspace of F. In the specific example we treated in the previous section, the arbitrariness $c \to c + \partial c'$ corresponds to a change in W space as $W \to W + \partial_F Y$ where $Y = c' \times I$. That is, the action $\int_W \eta_{DC}$ is unaffected if and only if

$$\int_{\partial_F Y} \eta_{DC} = \int_Y \mathrm{d}_F\eta_{DC} = 0 \tag{60}$$

where Y is a $(d+1)$-dimensional space. Let for example $c' = s_{1234}$. The above equation is guaranteed by the fact that continuity of the vielbein ensures that the pullbacks of $\mathrm{d}_{(t)}E(t)$ and $D(t)E(t)$ onto Y, with $E(t) = \sum_{i=1}^{4} t^i E_i$, vanish.

The composition rule (53) can be used to derive that the action $\int_W \eta(\omega(t), E(t))$ is formally equivalent to the action $\int_M \mathcal{L}(\omega, E)$ (Lemma 3 of Ref. [26]), provided there exists an everywhere continuous vielbein frame E and $DE = 0$. We prove that now in a more elegant and general way as follows.

Proposition 8. *The relation*

$$\phi^*\eta(\omega, E) = \phi^*\eta(\omega(t), E(t)) + \phi^*\mathrm{d}_F\mathcal{B} \tag{61}$$

holds, for some differential form $\mathcal{B}$, provided that conditions (i) *and* (ii) *are satisfied.*

Proof. We interpolate by $\omega(t,s) = (1-s)\omega(t) + s\omega$ and $E(t,s) = (1-s)E(t) + sE$, with $0 \le s \le 1$. We can easily show that $\phi^*(\mathrm{d}_F E(t,s)^a + \omega(t,s)^a{}_b E(t,s)^b) = 0$. By the Chern–Weil procedure one finds that (61) holds and

$$\mathcal{B} = n\int_0^1 \mathrm{d}s f((\omega - \omega(t)) \wedge \Omega_F(s)^{\wedge(n-1)} \wedge E(t,s)^{\wedge(d-2n)}) \tag{62}$$

where $\Omega_F(s) := \mathrm{d}_F\omega(t,s) + \frac{1}{2}[\omega(t,s), \omega(t,s)]$. We have then proved a gravitational version of the transgression formula (7). As ω is not well defined at the hypersurfaces, formula (61) holds in the weak sense.

We prove now the equivalence. We start by noting that:

$$\int_M \mathcal{L}(\omega, E) = \int_W \eta(\omega, E). \tag{63}$$

We integrate over W the identity of Proposition 8. From the Proposition B.1 we have that for $\partial M = 0$ the space W has no boundary so

$$\int_W \eta(\omega, E) = \int_W \eta(\omega(t), E(t)). \tag{64}$$

Therefore we have proved:

Proposition 9. *The action*

$$\mathcal{S} = \int_W \eta(\omega(t), E(t)) = \sum_i \int_i \mathcal{L}(\omega_i, E) + \sum_{k\ge 2}\frac{1}{k!}\sum_{i_1\ldots i_k}\int_{\{i_1\ldots i_k\}} \mathcal{L}(\omega_{i_1}, \ldots, \omega_{i_k}, E), \tag{65}$$

$$\mathcal{L}(\omega_0, \ldots, \omega_p, E) = \sum_{n=p}^{[d/2]} \alpha_n A_p \int_{s_{0\ldots p}} \mathrm{d}^p t\, f((\omega_1 - \omega_0) \wedge \cdots \wedge (\omega_p - \omega_0) \wedge \Omega(t)^{\wedge(n-p)} \wedge E^{\wedge(d-2n)}),$$

is formally equivalent to $\int_M \mathcal{L}(\omega, E)$.

By Proposition 7 the equations of motion (junction conditions) come only from variation with respect to the vielbein. Since the action is algebraic in the vielbein they are now easily obtained.

It is worth noting the following. The action $\mathcal{S}$ and the implied equations of motion involve explicitly only bulk data. One can easily prove the formula

$$\Omega(t) = \sum_i t^i \Omega_i - \frac{1}{4}\sum_{ij} t^i t^j [\omega_i - \omega_j, \omega_i - \omega_j]. \tag{66}$$

One only needs to calculate the bulk connection jumps and bulk curvature forms. The intrinsic connection on each hypersurface is virtually absent from the formulas. It is only implicitly there by continuity of the vielbein and the vanishing of torsion everywhere. If there is a discontinuity at the codimension one submanifold Σ, then the purely tangential part of the derived connections is continuous and defines an intrinsic connection of Σ. Consider a non-null Σ. Let an adapted frame (E^N, E^μ) at Σ where E^N is normal and E^μ are tangential to it. Then the pullback of the tangential torsion reads $i^*\mathrm{d}E^\mu + i^*\omega^\mu{}_\nu \wedge E^\nu = 0$. If E^μ is continuous across Σ then $i^*\omega^\mu{}_\nu$ is a natural intrinsic connection on it. The continuous components do not actively participate in the interpolations $\omega(t) = \sum_{i=0}^n t^i \omega_i$ i.e. it is absent in $\mathrm{d}_{(t)}\omega(t)$ and it cancels out in the differences $\omega_i - \omega_j$. Then, as we also discuss in more detail below, these bulk connection differences are simply the jump of the second fundamental form across the hypersurfaces ij. In general one need not explicitly introduce the Cauchy data of the hypersurface for the junction condition calculation. The results are equivalent and the actual calculations are often greatly facilitated.

Lack of need for intrinsic data in the junction conditions formulas means that they are applicable to the null hypersurface as well. This is also suggested by the fact that the inverse of the spacetime metric g appears nowhere. So if in particular the induced metric becomes degenerate somewhere, as in the case of null hypersurfaces, the formulas still hold.

Finally note that the vielbein may be more naturally given in a different frame on each side of the hypersurface. In this case, there is a set of vielbeins E^μ on one side and a different set of vielbeins $E^{\hat{\mu}}$ on the other. If the induced metric is the same then E^μ and $E^{\hat{\mu}}$ must be related by a local Lorentz transformation across a hypersurface. Then the connection also differs by a gauge transformation across the hypersurface. In order to get the correct junction conditions, this Lorentz transformation must be taken into account. In practice, it may be more convenient to do calculations on each side of the hypersurface with the respective natural intrinsic connection and put together the results. The justification of the calculation is based on the composition rule (8) with intrinsic as well as bulk connections involved, and it is an interesting problem on its own to be discussed elsewhere. In any case the result can be obtained from (65) by replacing E and the respective ω with the transformed ones.

6. Explicit junction conditions for intersections

Let us make contact with more standard formulations of junction conditions. We take the chance to comment on some quite remarkable qualitative differences compared to the Einstein gravity. We will consider non-null intersecting hypersurfaces.

The bulk field equation in terms of tensors takes the form:

$$\sum_n \beta_n H^\mu{}_\nu = T^\mu_\nu, \tag{67}$$

$$H^\mu{}_\nu := -2^{-n-1}\delta^{\mu\mu_1\cdots\mu_{2n}}_{\nu\nu_1\cdots\nu_{2n}} R^{\nu_1\nu_2}{}_{\mu_1\mu_2}\cdots R^{\nu_{2p-1}\nu_{2n}}{}_{\mu_{2n-1}\mu_{2n}} \tag{68}$$

where $H^\mu{}_\nu$ is the standard Lovelock tensor one obtains by varying (2) with respect to the metric.

First note that the variation of the bulk action with respect to the vielbein gives

$$\sum_n \beta_n \Omega^{b_1b_2}\wedge\cdots\wedge\Omega^{b_{2n-1}b_{2n}} e_{ab_1\ldots b_{2n}} = -2T^b{}_a e_b. \tag{69}$$

Above, it is convenient to define the rescaled constants $\beta_n = (d-2n)!\alpha_n$. The right hand side is defined so as to agree with (67) with $T^\mu_\nu = e^\mu_b e^a_\nu T^b_a$. We also introduced the following

$$e_{a_1\ldots a_k} = \frac{1}{(d-k)!}\epsilon_{a_1\ldots a_k a_{k+1}\ldots a_d} E^{a_{k+1}}\wedge\cdots\wedge E^{a_d}.$$

Note that $e = \sqrt{-g}$ is the volume element in d dimensions. The variation of this volume element with respect to the vielbein gives $\delta E^a e_a$. The intrinsic volume element on an intersection of codimension p is defined as follows: Let N_j, $j = 1,\ldots,p$ be orthonormal vectors, i.e. $N_i\cdot N_j = \eta_{ij}$ or $N_i\cdot N_j = \delta_{ij}$, on the normal space of the intersection $\{01\cdots p\}$. Let $E^a = (E^{N_1},\ldots,E^{N_p},E^\mu)$ be the adapted vielbein where E^μ spans the cotangent space of the intersection. We introduce the following differential forms:

$$\tilde{e}_{\mu\ldots\nu} := \prod_{i=1}^p (N_i\cdot N_i)\, e_{a_1\ldots a_p\mu\ldots\nu}(N_1)^{a_1}\cdots(N_p)^{a_p}. \tag{70}$$

In particular, $\tilde{e}$ is the induced volume element on the intersection and its variation with respect to the vielbein gives $\delta E^\mu \tilde{e}_\mu$.

On the simplicial intersection of bulk regions $\{0\}, \{1\}, \ldots, \{p\}$ the junction conditions read

$$(\mathcal{E}_{01\ldots p})_a = -2(T_{01\ldots p})_a^b \tilde{e}_b \tag{71}$$

where $(T_{01\ldots p})_a^b$ is the stress–energy tensor living on the intersection and $(\mathcal{E}_{01\ldots p})_a$ is simply the variation with respect to the vielbein of the surface term $\mathcal{L}(\omega_0, \ldots, \omega_p, E)$. It is given by:

$$\lambda^a \wedge (\mathcal{E}_{01\ldots p})_a \equiv \sum_{n=p}^{[d/2]} (d-2n)\alpha_n A_p \times \int_{s_{0\ldots p}} \mathrm{d}^p t\, i^* f((\omega_1 - \omega_0) \wedge \cdots \wedge (\omega_p - \omega_0) \wedge \Omega(t)^{\wedge(n-p)} \wedge \lambda \wedge E^{\wedge(d-2n-1)})$$

where λ is an arbitrary vector-valued 1-form. i^* is the pullback into the given intersection.

The second fundamental form of the hypersurface $\{ij\}$ embedded in the bulk region $\{i\}$ is [2]

$$\theta_{ij}^{ab} = i^*(\omega_i - \omega_\|)^{ab} = (N_{ij} \cdot N_{ij})(N_{ij}^a K_{ij}^b - N_{ij}^b K_{ij}^a) \tag{72}$$

where $\omega_\|$ is the intrinsic connection in $\{ij\}$ and i^* pulls the form back into this hypersurface. $N_{ji} = -N_{ij}$ by definition. $N_{ij} \cdot N_{ij} = \pm 1$. The 1-form K^a introduced is related to the extrinsic curvature tensor by $K^a := K^a{}_b E^b$. We will use the following convention: K_{ij}^a is the extrinsic curvature of the hypersurface $\{ij\}$ embedded in the bulk region $\{i\}$ (the first index), and

$$K_{ij}^{ab} = -h^{aa'} D_{a'} N_{ij}^b, \quad i > j; \qquad K_{ij}^{ab} = +h^{aa'} D_{a'} N_{ij}^b, \quad i < j. \tag{73}$$

Under this convention let us define

$$K_{[ij]}^a := K_{ij}^a - K_{ji}^a. \tag{74}$$

This is the jump of the extrinsic curvature across this hypersurface. Now consider a product of connection jumps $\omega_i - \omega_j$ as in (71) pulled back into an intersection. $\{ij\}$ is one of the hypersurfaces involved. As mentioned above the purely tangential components of the connection are continuous across hypersurfaces. Only the components $i^*(\omega_i - \omega_j)^{aN_{ij}}$ are non-zero, where i^* is the pullback into $\{ij\}$. By (72) we have that $i^*(\omega_i - \omega_j)^{ab} = (N_{ij} \cdot N_{ij})(N_{ij}^a K_{[ij]}^b - N_{ij}^b K_{[ij]}^a)$.

After some calculation, using the identity

$$i^* E^{\nu_1 \ldots \nu_n} \wedge \tilde{e}_{\mu_1 \ldots \mu_m} = \frac{m!}{(m-n)!} \delta^{\nu_1}_{[\mu_{m-n+1}} \cdots \delta^{\nu_n}_{\mu_m} \tilde{e}_{\mu_1 \ldots \mu_{m-n}]},$$

we find that the junction condition is:

$$(T_{01\ldots p})^\tau_\sigma = -\sum_{n=p}^{[d/2]} \varsigma_{0\ldots p} \beta_n\, 2^{p-1} \frac{n!}{(n-p)!} (2n+1-p)! \times \det(M^j{}_i) \int_{s_{01\ldots p}} \mathrm{d}^p t\, (K_{[10]})^{\nu_1}_{[\nu_1} \cdots (K_{[p0]})^{\nu_p}_{\nu_p} \, \Omega(t)^{\nu_{p+1}\ldots\nu_{2n-p}}_{\nu_{p+1}\ldots\nu_{2n-p}} \delta^\tau_{\sigma]} \tag{75}$$

where

$$\varsigma_{0\ldots p} := \prod_{i=1}^p N_{i0} \cdot N_{i0} \prod_{j=1}^p N_j \cdot N_j$$

is $+1$ for a spacetimelike intersection and ± 1 for a spacelike intersection depending on the arrangement of the hypersurfaces which meet there. Also we have defined the matrix $M^j{}_i$ by

$$(N_{i0})^a = \sum_{i=1}^p M^j{}_i\, N_j^a.$$

Compact notation of the form $\Omega^{\nu_1\ldots\nu_{2k}}_{\nu_1\ldots\nu_{2k}}$ standing for $\Omega^{\nu_1\nu_2}_{[\nu_1\nu_2} \ldots \Omega^{\nu_{2k-1}\nu_{2k}}_{\nu_{2k-1}\nu_{2k}]}$ will be used for convenience.

The curvature $\Omega(t)$ can be expressed in terms of the curvatures and the connections of the individual regions by the symmetrical formula (66). One may write the purely tangential components of this curvature in terms of curvature of the bulk regions and quadratic terms in the extrinsic curvatures.

$$\Omega(t)^{\mu\nu}{}_{\kappa\sigma} = \frac{1}{2}\left(\sum_i t^i (R_i)^{\mu\nu}_{\kappa\sigma} + \sum_{i>j} t^i t^j (N_{ij}\cdot N_{ij})(K_{[ij]}{}^{\mu}{}_{\kappa} K_{[ij]}{}^{\nu}{}_{\sigma} - K_{[ij]}{}^{\mu}{}_{\sigma} K_{[ij]}{}^{\nu}{}_{\kappa})\right) \tag{76}$$

where $(R_i)^{\mu\nu}_{\kappa\sigma}$ are the tangential components of the Riemann tensor in the bulk region $\{i\}$, pulled back onto the intersection.

Eqs. (75) and (76) give the building blocks for writing the junction conditions for any non-null intersection. The most explicit formula for the junction conditions involves integrating over the simplex. This can always be done using the integrals

$$\int_{s_{01\ldots p}} \mathrm{d}^p t t_0^{n_0}\ldots t_p^{n_p} = \frac{n_0!\ldots n_p!}{\left(p+\sum\limits_{i=0}^{p} n_i\right)!}$$

but the final expression can be rather involved.

Let us now look at some of the important differences between Einstein gravity and higher order Lovelock gravity. Consider a single hypersurface separating bulk regions labelled by 0 and 1. Unlike Einstein gravity (Israel junction condition) a vanishing (non-null) hypersurface's energy tensor

$$T_{01} = 0 \tag{77}$$

does not imply continuity of the connection (that is, zero jump of the extrinsic curvature) in Lovelock gravity: the relevant Lagrangian involves polynomial terms of the extrinsic curvatures and of the intrinsic curvature of the hypersurface and the various terms may well cancel each other. In general, one cannot deduce an explicit expression for the jump of the extrinsic curvature from T_{ij} as one can do in Einstein case.

Let us see this difference explicitly. First, consider Einstein gravity with cosmological constant, that is, only β_1 and β_0 are non-zero. For the bulk we have

$$G_a^b = \frac{1}{\beta_1} T_a^b + \frac{\beta_0}{2\beta_1}\delta_a^b \tag{78}$$

where G_a^b is the Einstein's tensor. The junction conditions for the hypersurface read

$$(K_{[10]})^{\tau}_{\sigma} - K_{[10]}\,\delta^{\tau}_{\sigma} = \frac{1}{\beta_1}(T_{01})^{\tau}_{\sigma}. \tag{79}$$

As is well known, $T_{01} = 0$ if and only if $(K_{[10]})^{\tau}_{\sigma} = 0$.

In the general case, the junction conditions for the hypersurface are

$$\sum_{n=1}^{[d/2]} \beta_n\,(2n)!\,n \int_0^1 \mathrm{d}t\,(K_{[10]})^{\nu_2}_{[\nu_2}\,\Omega(t)^{\nu_3\ldots\nu_{2n}}_{\nu_3\ldots\nu_{2n}}\delta^{\tau}_{\sigma]} = -(T_{01})^{\tau}_{\sigma}. \tag{80}$$

Vanishing of the energy tensor T_{01} does *not* imply that the extrinsic curvature is continuous across the hypersurface in Lovelock gravity.

Let the discontinuity $K_{[10]}$ be infinitesimal. Then the junction condition reads

$$\sum_{n=1}^{[d/2]} \beta_n\,2^{-n+1}(2n)!\,n\,(K_{[10]})^{\nu_2}_{[\nu_2}\,R^{\nu_3\ldots\nu_{2n}}_{\nu_3\ldots\nu_{2n}}\delta^{\tau}_{\sigma]} = -(T_{01})^{\tau}_{\sigma} \tag{81}$$

to first order in $K_{[10]}$. Then $T_{01} = 0$ for an arbitrary infinitesimal $K_{[10]}$ if and only if the quantity

$$\sum_{n=1}^{[d/2]} \beta_n\,2^{-n+1}\,(2n)!\,n\,\delta^{\nu}_{[\mu}\delta^{\tau}{}_{\sigma}R^{\nu_3\ldots\nu_{2n}}_{\nu_3\ldots\nu_{2n}]} \tag{82}$$

vanishes. Inversely, we may think of it as a matrix $M_{IJ} \equiv M_{(\mu\nu)(\sigma\tau)}$. If the Cauchy data evolve under the condition that det $M_{IJ} \neq 0$, then (infinitesimal) $K_{[10]} = 0$ if and only if $T_{01} = 0$, that is, $K_{[10]} \neq 0$ if and only if $T_{01} \neq 0$ so that the extrinsic curvature cannot get a discontinuity across a hypersurface without a T_{01}. This is Choquet-Bruhat's condition for a well posed initial value problem in Lovelock gravity [33]. When this determinant becomes zero, there is a breakdown of predictability in the theory [20,18], which is a key problem to be addressed if Lovelock gravity is to be regarded as a physical theory of gravity.

The next more complicated thing than the hypersurface is the codimension 2 intersection. The simplest case comes from the Gauss–Bonnet term and reads

$$2X^{\tau}_{\sigma} - \delta^{\tau}_{\sigma} X^{\rho}_{\rho} = (T_{012})^{\tau}_{\sigma} \tag{83}$$
$$X^{\tau}_{\sigma} \equiv 4\beta_2 \det(M^{j}_{i})((K_{[10]})^{\rho}_{\rho}(K_{[20]})^{\tau}_{\sigma} + (K_{[20]})^{\rho}_{\rho}(K_{[10]})^{\tau}_{\sigma} - 2(K_{[10]})^{\rho}_{\sigma}(K_{[20]})^{\tau}_{\rho}).$$

Note first that the energy tensor T_{012} on the intersection vanishes if and only if the matrix X^{τ}_{σ} vanishes. This does not imply that the jumps $K_{[10]}$ and $K_{[20]}$ vanish too. On the other hand, intersecting or colliding shells of matter will in general produce a non-zero energy tensor on the hypersurface where their spacetime trajectories meet. This is unavoidable however small β_2 might be.

The most obvious physical implication coming out of this work and the previous one [26], is that intersections in Lovelock gravity involves hypersurfaces of various codimensionalities carrying non-zero energy tensors. In particular one can consider a collision of shells, that is, a spacelike intersection C of timelike hypersurfaces, with a total of m ingoing and outgoing shells. There could exist a non-zero stress tensor on C, T_C. (The appearance of a stress tensor on the spacelike surface would have to be due to some exotic kind of matter, in conflict with the dominant energy condition [34] as mentioned already in [26].) An implication is that we may have m outgoing or m ingoing shells i.e. the shells of matter may all originate from or disappear into C. This also is forbidden in Einstein gravity by the conservation of energy for positive energy densities, but perfectly possible here because of T_C (see Section 3.2 of [26] for the energy exchange relations). The reason is that the energy exchange relations involve also extrinsic curvatures and C acts as *source* which can emit or absorb all m timelike trajectories. The dominant energy condition is respected in Lovelock gravity collisions if the bulk geometries are constraints by the condition $T_C = 0$.

We have mentioned that hypersurfaces with no energy tensor are possible in Lovelock gravity. That is a discontinuity of the connection can be self-supported. We showed in Ref. [35] that one may have a bulk AdS spacetime vacuum with such discontinuities. More complicated cases do not seem impossible. If perfect homogeneity is given up, we have an interesting kind of vacuum in this gravity.

Continuing with matters of vacuum, thin shells and their gravitational effects appear when separating the phases of false and true vacuum in false vacuum decay in the presence of gravity [36]. This happens when there are more one (local) minima of the energy of a system and not all of them have the same value. A (only) local minimum state, false vacuum, decays by formation of bubbles of true vacuum which grow very fast and eventually collide. The bubble effects play an important role in the inflationary evolution model of the early universe [37], the implications of collisions have been studied in [38]. In a 'universe' with more than four dimensions at those times or in general, the collisions of the bubbles have additional effects as we have learned here: at the spacelike hypersurface of collision there will in general live a non-zero stress tensor. That is, an instanton-like configuration of non-topological nature. These may have interesting implications, they are though beyond the scope of this paper.

7. Conclusion

The theory of General Relativity (GR) admits singular sources whose stress–energy tensor has support on a hypersurface. In general, an arbitrary collection of such objects should intersect. We have shown that in gravities including dimensionally continued Euler densities with up to n factors of the Riemann tensor, not only are such hypersurface sources well defined but that there is a possibility of sources of codimension up to n at the intersections. This becomes possible because the equations of motion are not linear in curvature and of the fully antisymmetric way the Riemann tensors are contracted. To give an example, imagine we have two hypersurfaces crossing each other, one at $x = 0$ and the other where $y = 0$. Then some components of the curvature will have a $\delta(x)$ singularity and some other will have a $\delta(y)$ one. If the theory of gravity includes a Gauss–Bonnet term, the energy tensor will have a term of the form $\delta(x)\delta(y)$ i.e. a codimension two matter distribution localized at $x = y = 0$. In Einstein theory if the curvature is of that form one could not have a codimension two matter. Schematically,

$$f(\Omega \wedge E^{\wedge(d-2)}) \approx A\delta(x) + B\delta(y),$$
$$f(\Omega^{\wedge 2} \wedge E^{\wedge(d-4)}) \approx C\delta(x, y).$$

A $\delta(x, y)$ could be produced in Einstein gravity only if the curvature itself had such a singularity, which implies a conical singularity.

The $n = 1$ (GR) junction conditions give a 1-1 correspondence between the discontinuity of the connection and the energy–momentum tensor on a hypersurface. For the Lovelock theory with higher n terms, things are more complicated: the energy–momentum tensor is a polynomial in the curvature and discontinuity of the connection. For a singular energy–momentum tensor to be supported, there must be a discontinuity, but the converse does not apply. It is possible for the energy–momentum tensor to vanish even if there is a discontinuity.

The metric describing an intersection of hypersurfaces which in GR has no localized matter at the intersection, will generally produce localized matter due to the non-trivial junction conditions for the higher order Lovelock terms. 'Intersection' is a general term and includes the case where the intersection hypersurface is space-like where we have a collision. Then, if we demand no localized space-like matter there will be a constraint on the geometry. The constraint will be of order α_2, the coefficient of the quadratic Lovelock term. Thus, the higher order Lovelock terms place additional constraints on the way that singular matter sources can interact with each other. This qualitative difference is well illustrated by a planar intersection in AdS space [35].

Expressions like $P\left([\mathrm{d}_{(t)}A(t) + F(t)]^{\wedge n}\right)$, descended from a Characteristic Class $P\left(F^{\wedge n}\right)$, are already known in the mathematics literature [32] and in the context of anomalies in gauge theory or gravity [39]. The homotopy operator (34) has appeared in Ref. [40]. We have shown that these geometrical methods are very useful when studying intersections of hypersurfaces in gravity theories.

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Appendix A. Proof of $\mathrm{d}_F\eta = 0$

Recall the definition of $\Omega(t)$ and also the Jacobi identity [41]:

$$\Omega(t) = \mathrm{d}_{(x)}\omega(t) + \frac{1}{2}[\omega(t), \omega(t)] \tag{A.1}$$

$$[[\omega, \omega], \omega] = 0. \tag{A.2}$$

From these one can easily find the following identities.

$$\mathrm{d}_{(x)}[\omega(t), \omega(t)] = 2[\Omega, \omega(t)] \tag{A.3}$$

$$\mathrm{d}_{(t)}[\omega(t), \omega(t)] = 2[\mathrm{d}_{(t)}\omega(t), \omega(t)] \tag{A.4}$$

$$\mathrm{d}_{(t)}\omega(t) + \Omega(t) = \mathrm{d}_F\omega(t) + \frac{1}{2}[\omega(t), \omega(t)]. \tag{A.5}$$

Also, from (A.3), we get the Bianchi identity for $\omega(t)$:

$$D(t)\Omega(t) = 0. \tag{A.6}$$

Like ω_0, $\omega(t)$ is a connection and so the invariance property of f implies, for 2-forms ψ:

$$\sum_i f(\psi_1 \wedge \cdots [\omega(t), \psi_i] \cdots \wedge \psi_n) = 0. \tag{A.7}$$

Combining (A.4)–(A.6):

$$\mathrm{d}_F(\mathrm{d}_{(t)}\omega(t) + \Omega(t)) = [\mathrm{d}_{(t)}\omega(t) + \Omega(t), \omega(t)] \tag{A.8}$$

which is equivalent to (30) and so our Proposition 2 follows by the invariance property of the Polynomial (A.7).

$$\mathrm{d}_F f\left([\mathrm{d}_{(t)}\omega(t)+\Omega(t)]^{\wedge n}\right)=0. \tag{A.9}$$

Let us expand the polynomial:

$$\begin{aligned}[\mathrm{d}_{(t)}\omega(t)+\Omega(t)]^{\wedge n} &= \sum_{l=0}^{n} {}^{n}C_l \left(\sum_{\alpha=1}^{p}\mathrm{d}_{(t)}\omega(t)\right)^{\wedge l}\wedge\Omega(t)^{\wedge(n-l)}\\ &= \sum_{l=0}^{n}(-1)^{l(l-1)/2}\,{}^{n}C_l\ \mathrm{d}t^{i_1}\wedge\cdots\wedge\mathrm{d}t^{i_l}\\ &\quad \wedge\mathrm{d}_{(t)}\omega(t)_{i_1}\wedge\cdots\wedge\mathrm{d}_{(t)}\omega(t)_{i_l}\wedge\Omega(t)^{\wedge(n-l)}.\end{aligned} \tag{A.10}$$

The first term in the expansion evaluated at the 0-simplex s_i is just the Euler density (6) in the interior of the region i. Thus (A.9), combined with (25) completes the proof by induction of the second proposition. As a consistency check, we can see that the terms in this expansion reproduce the form of (19).

Appendix B. W-space and topology

The closure of η in F but also means that it obeys the same transgression formula as the invariant polynomial we started with, only now on F. Under continuous variation $\omega(t)\to\omega'(t)$,

$$P(\Omega_F)-P(\Omega'_F)=\mathrm{d}_F TP(\omega(t),\omega'(t)). \tag{B.1}$$

Let us for now assume that M is compact. We define a covering of open sets on W by the open sets on M. Choose a covering of M. For every open $O_i\subset M$ define the set $\mathcal{O}_i\subset W$ as the set $\{(t,x)\in W|x\in O_i\}$. Clearly $\mathcal{O}_i$'s cover W and are open sets,[3] endowing W with a manifold structure. This gives W the topology of M. For a partition of unity f_i of M we define the partition of unity of W simply by $f_i(t,x)=f_i(x)$. Then, by the invariance (C.1), (31) is meaningful over W associated with a topologically non-trivial M just as $\int_M\mathcal{L}(\omega)$ is meaningful over M.

The shape of W is interesting. Every $d-1$ dimensional surface is thickened in the t-direction by a 1-dimensional simplex; These meet at a $d-2$ surface in M which looks like a triangular prism in W (Fig. 1(c)), etc. We know that the equality holds:

$$\int_M\mathcal{L}(\omega)=\int_W\eta(\omega(t))\quad(\propto\text{ Euler no.}). \tag{B.2}$$

All that we did in Section 3 amounts to expanding both sides via (9) and (A.10), and equating the terms. Given that M and W have the same topology (and Euler number) we can say that $\eta(\omega(t))$ is the Euler density of W.

If we calculate $\int_M\mathcal{L}(\omega)$ with a different C^0 metric[4] (with discontinuities of the connection at intersecting hypersurfaces) described by an $\omega'(t)$, we have along with (B.2) the relation

$$\int_M\mathcal{L}(\omega)=\int_W\eta(\omega'(t)). \tag{B.3}$$

Then (B.1) tells us that

$$\int_W\mathrm{d}_F TP(\omega(t),\omega'(t))=0. \tag{B.4}$$

[3] One may face problems only if an infinite number of intersections is considered, locally. Apart from that, the t-excursions add no accumulation points, that is boundary points, to the sets O_i.

[4] The argument of $\mathcal{L}$, ω, is unrelated to $\omega(t)$; they are just both assumed to produce the same Euler number. As we explained in [26], ω is associated with a smooth metric, say C^2. Our formulas give the wanted independence from the connection, of certain topological quantities, when discontinuities are allowed.

This is true for quite arbitrary $\omega(t)$, $\omega'(t)$ so we must have

$$\partial_F W = 0. \tag{B.5}$$

Proposition B.1. *Define $\pi : W \to M$, $\pi(x,t) = x$. Then $\pi(\partial_F W) \subset \partial M$ if and only if:*

$$\partial\{i_0 \ldots i_p\} = \sum_{i_{p+1}} \{i_0..i_p i_{p+1}\} + \{i_0 \ldots i_p\} \cap \partial M.$$

In particular $\partial_F W = 0$ if $\partial M = 0$. These relations can be taken as the definition of the simplicial intersections, which we used in Ref. [26].

Proof. When each ω_i, E_i are chosen to be that of each bulk region i, then:

$$W = \sum_{p=0}^{h} \sum_{i_0 \ldots i_p} \frac{1}{(p+1)!} s_{i_0 \ldots i_p} \times \{i_0 \ldots i_p\} \subset F \tag{B.6}$$

where $\{i_0 \ldots i_p\} \subset M$ is a codimension p submanifold and as a point-set corresponds to the codimension p simplicial intersection. h is the codimension of the highest codimension intersection present. Note that it is sufficient to take $\{i_0 \ldots i_p\}$ fully antisymmetric.

Then,

$$\partial_F W = \sum_{p=0}^{h} \sum_{i_0 \ldots i_p} \frac{1}{(p+1)!} \left\{ \partial s_{i_0 \ldots i_p} \times \{i_0 \ldots i_p\} + (-1)^p s_{i_0 \ldots i_p} \times \partial\{i_0 \ldots i_p\} \right\},$$

where $\partial_{(x)} s_p = (-1)^p s_p \partial_{(x)}$ was used, and $\partial_F = \partial_{(x)} + \partial_{(t)}$. For the first term we have

$$\begin{aligned} \sum_{p=0}^{h} \sum_{i_0 \ldots i_p} \frac{1}{(p+1)!} \sum_{r=0}^{p} (-1)^r s_{i_0 \ldots \hat{i}_r \ldots i_p} \times \{i_0 \ldots i_p\} &= \sum_{p=1}^{h} \sum_{i_0 \ldots i_p} \frac{1}{(p+1)!} (p+1)(-1)^p s_{i_0 \ldots i_{p-1}} \times \{i_0 \ldots i_p\} \\ &= \sum_{p=0}^{h-1} \sum_{i_0 \ldots i_p i_{p+1}} \frac{1}{(p+1)!} (-1)^{p+1} s_{i_0 \ldots i_p} \times \{i_0 \ldots i_p i_{p+1}\} \end{aligned} \tag{B.7}$$

so combining with the second term we have

$$\begin{aligned} \partial_F W = &\sum_{p=0}^{h-1} \sum_{i_0 \ldots i_p} (-1)^p \frac{1}{(p+1)!} s_{i_0 \ldots i_p} \times \left\{ -\sum_{i_{p+1}} \{i_0 \ldots i_p i_{p+1}\} + \partial\{i_0 \ldots i_p\} \right\} \\ &+ (-1)^h s_{01 \ldots h} \times \partial\{01 \ldots h\}. \end{aligned} \tag{B.8}$$

For the highest codimension surface one has

$$\partial\{01 \ldots h\} = \{01 \ldots h\} \cap \partial M. \tag{B.9}$$

Thus we get Proposition B.1. If we ignore boundary terms on ∂M, we may justly ignore boundary terms on $\partial_F W$.

Appendix C. Invariance of η

In this appendix it is shown that the η-form is invariant under gauge transformations:

$$\eta(\omega(t)) = \eta(\omega(t)_{(g)}). \tag{C.1}$$

Under the change $\omega_i \to \omega_{i\,(g)}$ of the connection of every region $\{i\}$ with

$$\omega_{i\,(g)} = g^{-1} \omega_i g + g^{-1} \mathrm{d}_{(x)} g. \tag{C.2}$$

The interpolating connection $\omega(t) = \sum_{i=0}^{p} t^i \omega_i$ (with $\sum_{i=0}^{p} t^i = 1$) changes as $\omega(t) \to \omega(t)_{(g)}$ with

$$\omega(t)_{(g)} = g^{-1}\omega(t)g + g^{-1}\mathrm{d}_{(x)}g. \tag{C.3}$$

Then both

$$\mathrm{d}_{(t)}\omega(t) = \sum_{i=1}^{p} \mathrm{d}t^i(\omega_i - \omega_0), \qquad \Omega(t) = \mathrm{d}_{(x)}\omega(t) + \frac{1}{2}[\omega(t), \omega(t)] \tag{C.4}$$

obviously transform as

$$\mathrm{d}_{(t)}\omega(t)_{(g)} = g^{-1}\mathrm{d}_{(t)}\omega(t)g, \qquad \Omega(t)_{(g)} = g^{-1}\Omega(t)g \tag{C.5}$$

so Ω_F itself changes to

$$(\Omega_F)_{(g)} = g^{-1}\Omega_F g \tag{C.6}$$

and the invariant polynomial P gives us the wanted invariance relation (C.1).

Actually, the abstraction of Ω_F as a curvature associated to $\omega(t)$ and the derivative operator d_F helps us again to prove things easily. By its very definition (which let us repeat)

$$\Omega_F = \mathrm{d}_F\omega(t) + \frac{1}{2}[\omega(t), \omega(t)]$$

we see that under $\omega(t) \to \omega(t)_{(g)}$ with

$$\omega(t)_{(g)} = g^{-1}\omega(t)g + g^{-1}\mathrm{d}_F g \tag{C.7}$$

we have immediately the covariant transformation (C.6). But $\mathrm{d}_F g = \mathrm{d}_{(x)}g$ so we proved again (C.1).

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